Are Time-Varying Systems Laplace-Transformable?

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Outline of the Presentation

- Objectives
- From LTV Elements to LTV Systems
- Observations on LTV Systems
  - Bivariate Time and Bifrequency Characterization
  - Generalized-Delay System Representation
  - Circularly Symmetric Functions
- Two-Dimensional Laplace Transform (2DLT)
  - Laplace –Carson Transform
  - Transformation of Nonanticipative Systems
  - Transformation of Synchronized Systems
- Conclusions
- References

Objectives

- There has been an increasing interest in realization and implementation of linear time-varying (LTV) and adaptive systems.
- The rather limited use of LTV networks in analog signal processing and communication fields is due largely to lack of powerful means for their analysis, synthesis and performance evaluation.
- The theory of LTV systems has been widely based on the time- domain approach.
- The main objective of this project is to provide a unified treatment for the analysis and synthesis of LTV systems.
- The approach to be described consists of extension of Laplace transform techniques commonly used for linear time-invariant (LTI) systems.

Fundamental Input-Output Representation

- **The classical theory of variable systems is based on the solutions of linear ordinary differential equations with varying coefficients.**
  - **The varying coefficients are functions of an independent variable, conveniently called the *time*.**
  - **The *time* is assumed to be *real* for physical systems.**

$$\sum_{i=0}^{n} a_i(t)\, y^{(i)}(t) = \sum_{k=0}^{m} b_k(t)\, x^{(k)}(t)$$

Observations on LTV Dynamic systems

Observation 1 – In general, time variables of the *signal* and *system* do not have to be synchronized; i.e., the (time) variables of the signal and system are *independent* of each other.

$$L(D,\tau)y(t) = K(D,\tau)x(t)$$

Observation 2 – At any instant of *"t"* there is a response, which is a specified function of "τ".

Observation 3 – At any fixed "*τ*" there is a response, which is a specified function of "t".

Observation 4 – The system response is a function of variations of *observation* parameter "t" and *application* parameter "*τ*".

Observation 5 – A zero-input, SISO LTV system described by:

$$L(D,\tau)y(\cdot) = 0$$

is a linear system that its natural frequencies are varying with "*τ*". In other words, solutions of this equation are exponential functions of time with varying natural frequencies, as given by:

$$y(\cdot) = \sum_{i=0}^{n} c_i e^{-t\alpha_i(\tau)}$$

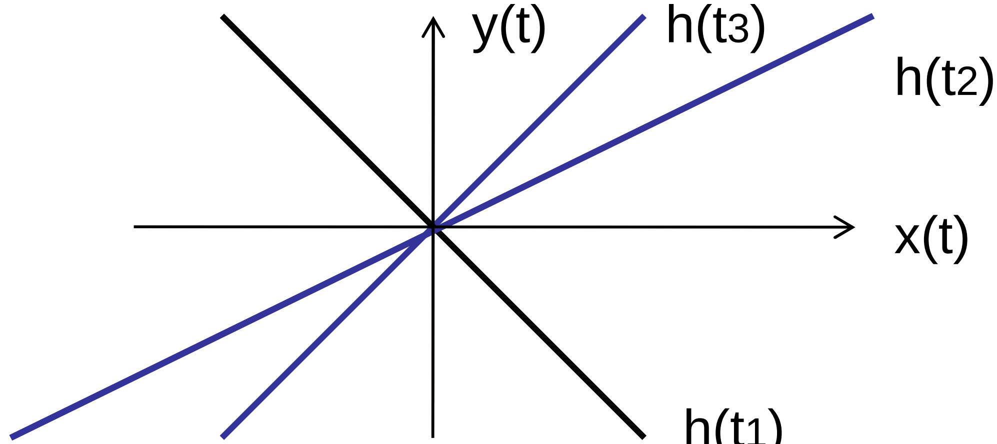
where $\alpha_i(\tau)$ is a function of variable coefficients of the fundamental equation of the system under consideration.

Linear Time Varying Elements

- **A single-input single-output (SISO) dynamic system element of finite order characterized by its input-output relationship is said to be linear if the following holds for each t ≥0:**

$$y(t) = h(t)x(t)$$

- **Where *h(t)* is the system function defines the response at time t, denotes the slope of the y-x curve in a rectangular coordinates system.**



Linear Time Varying Operator

- **A SISO dynamic system operation is shown symbolically by:**

$$y(t) = \mathcal{O}\{x(t)\}$$

- **The system operator is linear if and only if the following relation holds:**

$$\mathcal{O}\{\alpha x_1(t) + \beta x_2(t)\} = \alpha \mathcal{O}\{x_1(t)\} + \beta \mathcal{O}\{x_2(t)\} = \alpha y_1(t) + \beta y_2(t)$$

- **The system input can be *any* function including an impulse or a *delta function:***

$$y_\delta(t;\tau) = h(t)\delta(t-\tau)$$

Observing an Impulse Response Function (1)

- **Observation 6 – The following symbolic identity holds:**

$$h(t)\delta(t-\tau)=h(\tau)\delta(\tau-t)$$

- **Observation 7 - The product** $h(t)\delta(t-\tau)$ **is different from zero at the point** $t=\tau$**.**

- **Observation 8 - The impulse response of the system has a circular symmetric property with respect to its arguments** t **and** τ**.**

$$y_\delta(t;\tau)=y_\delta(\tau;t)$$

Observing an Impulse Response Function (2)

- **Observation 9 – In the *(t,τ)-plane*, due to the circular symmetry and because delta function is an even function, we can define a bivariate response function as:**

$$y(t;\tau)=h(t,\tau)x(\tau,t)\delta(|t-\tau|)$$

- **Observation 10 - The ordinary output response at the point *t=τ* *is:***

$$y(\tau)=\int_{0_-}^{+\infty} h(t,\tau)x(\tau,t)\delta(|t-\tau|)dt=h(\tau)x(\tau)$$

or, equivalently:

$$y(t)=\int_{0_-}^{+\infty} h(t,\tau)x(\tau,t)\delta(|\tau-t|)d\tau=h(t)x(t)$$

Question – Can a system function be equal to an input function?

Is the system output considered to be still linear?

Frequency-Domain Characterization

- **A linear time-invariant dynamic system described by a homogeneous nth-order differential equation:**

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = 0$$

- **Using the operator *d/dt* → *s*, this equation can be written as:**

$$\sum_{i=0}^{n} a_i s^i y(t) = 0$$

- **Thus, the homogenous solution is a linear combination of exponentials:**

$$y(t) = \sum_{i=0}^{n} a_i e^{-s_i t}$$

Roots of Characteristics Operator Equation

Frequency-Domain Characterization (Cont.)

- **A linear time-invariant dynamic system with the applied input *x(t)*:**

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} x(t)$$

- **The system is turned on at *t*, the impulse-response at *t-τ* is obtained:**

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t;\tau) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k} \delta(t-\tau)$$

- **We observe:**
  - **The response is a bivariate function of *t* and *τ***
  - **Characteristics roots are a function of *τ***

$$y(t,\tau) = \sum_{i=0}^{n} a_i e^{-s_i(\tau)t}$$

Roots of Characteristics Operator Equation

Laplace Transform of the Impulse Function

- **The ordinary unilateral Laplace transform of $\delta(t-\tau)$ is obtained as:**

$$L\{\delta(t-\tau)\} = \int_{0-}^{+\infty} \delta(t-\tau) e^{-s_1 t} dt = e^{-s_1 \tau}$$

- **This is a function of the variable application time *τ*.**

- **A second transformation yields:**

2DLT

$$L_{2D}\{\delta(t-\tau)\} = \int_{0-}^{+\infty} e^{-s_1 \tau} e^{-s_2 \tau} d\tau = \frac{1}{s_1 + s_2}$$

Laplace-Carson Transform

Definition of 2DLT – The ordinary unilateral 2DLT is defined as:

$$\boldsymbol{H}(\boldsymbol{s}_1,\boldsymbol{s}_2)=\int_0^{+\infty}\int_0^{+\infty}\boldsymbol{h}(\boldsymbol{t}_1,\boldsymbol{t}_2)\boldsymbol{e}^{-s_1t_1}\boldsymbol{e}^{-s_2t_2}\boldsymbol{dt}_1\boldsymbol{dt}_2$$

Inverse Transformation - The inverse 2DLT is given by:

$$L_{2D}^{-1}\{H(s_1,s_2)\}=h(t_1,t_2)$$

$$=\frac{1}{(2\pi j)^2}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty}\int_{\sigma_1-j\infty}^{\sigma_1+j\infty}H(s_1,s_2)e^{s_1t_1}e^{s_2t_2}ds_1ds_2$$

Two-Dimensional Laplace Transform (2DLT)

Observation 11– For conformal transformation, it is required that the unit function u(t, τ) transforms into itself:

$$u(t,\tau) \Leftrightarrow U(s_1, s_2) = 1$$

u (t,τ) is equal to 1 when both t and τ are positive, and is equal to zero when at least one of the arguments is negative.

Observation 12- Based on the above observation we modify the 2DLT is given by:

$$h(t,\tau) \Leftrightarrow H(s_1, s_2) = s_1 s_2 \int_0^{+\infty}\int_0^{+\infty} h(t,\tau) e^{-s_1 t} e^{-s_2 \tau} dt d\tau$$

Laplace-Carson Transform

Two-Dimensional Step Function

- **The unit–step function u(t, τ) is defined as:**

$$u(t,\tau)=u(t)u(\tau)$$

The Laplace-Carson transform of u(t-τ) is

$$U(s_1,s_2)=s_1s_2\int_0^{+\infty}\int_0^{+\infty}u(t-\tau)e^{-s_1t}e^{-s_2\tau}dtd\tau=\frac{s_2}{s_1+s_2}$$

Similarly, the L-C transform of u(τ-t) is $\frac{s_1}{s_1+s_2}$

Observation 13 **- We can write:**

$$u(t-\tau)u(\tau-t)=\begin{cases}1 & t=\tau\\ 0 & t\neq\tau\end{cases}$$

The 2DLT transform of $u(t-\tau)u(\tau-t)$ **is** $\frac{1}{s^2}$ **.**

Two-Dimensional Impulse Function

- The unit–impulse function δ(t, τ) is defined as:

$$\delta(t,\tau)=\delta(t)\delta(\tau)$$

The Laplace-Carson transform of δ(t-τ) is

$$\Delta(s_1,s_2)=s_1s_2\int_0^{+\infty}\int_0^{+\infty}\delta(t-\tau)e^{-s_1t}e^{-s_2\tau}dt\,d\tau=\frac{s_1s_2}{s_1+s_2}$$

Similarly, the L-C transform of δ(t,τ) is $\delta(t,\tau)\Leftrightarrow s_1s_2$

Observation 14 - **We can obtain the L-C transform of** $h(t)\delta(t-\tau)$ **as:**

$$h(t)\delta(t-\tau)\Leftrightarrow s_1s_2\int_0^{+\infty}h(\tau)e^{-(s_1+s_2)\tau}d\tau=s_1s_2H(s_1+s_2)$$

Impulse-Response System Representation

- **The system response for a circularly symmetric system function can equivalently be written as:**

$$y_{\delta}(t,\tau) = h(t,\tau)\delta(t-\tau)$$

- **The more familiar impulse response, using *sifting property* of the *delta function* will be:**

$$y_{\delta}(\tau) = \int_{t=\tau_-}^{t=\tau_+} h(t,\tau)\delta(t-\tau)dt = h(\tau)$$

➢ **The limits of integration can be extended to infinity.**

Nonanticipative System Function

- Define the instant at which the input is applied to the system as the origin for time *"t."*

- The nonanticipative condition implies:

$$h(t-\tau)x(t)u(t) \equiv o \qquad for \quad t < \tau$$

$$h(t-\tau)x(t)u(t) \equiv y(t,\tau) \qquad for \quad t > \tau$$

- Then, we may define *h(.)* to be zero for negative values of its argument:

Frequency-Domain Representation of Nonanticipative System Functions

- **Let us define:**

$$h_1(t,\tau)=\begin{cases} h(t-\tau) & for \quad t>\tau \\ 0 & for \quad t<\tau \end{cases}$$

- **The 2DLT is:**

$$H_1(s_1,s_2)=\int_0^{+\infty} e^{-s_2\tau}d\tau\int_\tau^{+\infty} e^{-s_1 t}h(t-\tau)dt = H(s_1)/(s_1+s_2)$$

- **Similarly, we define :**

$$h_2(t,\tau)=\begin{cases} h(\tau-t) & for \quad \tau>t \\ 0 & for \quad \tau<t \end{cases}$$

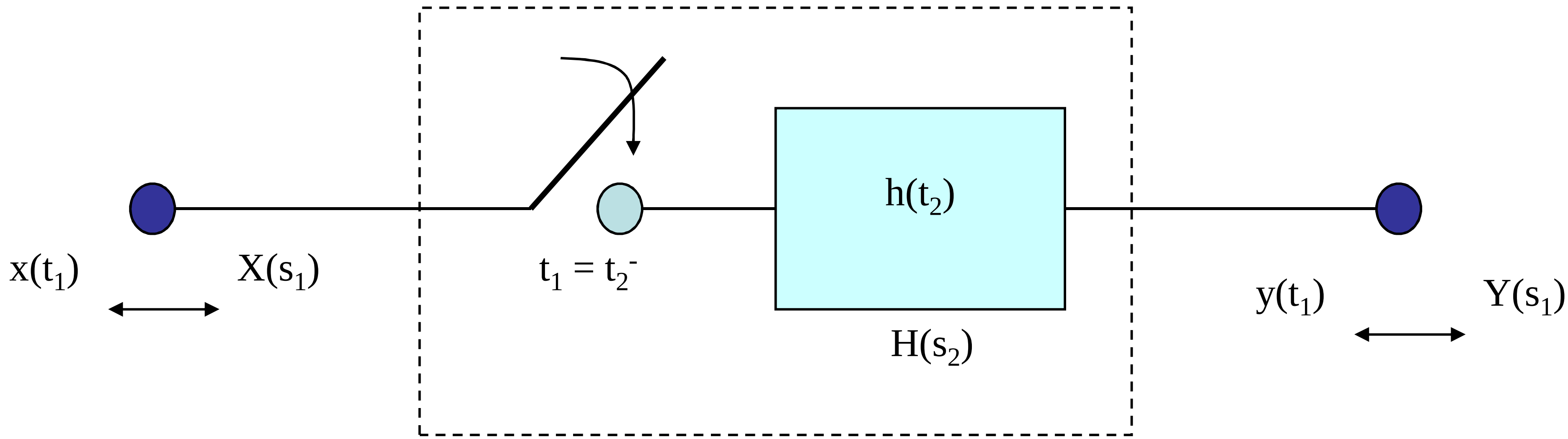
$$H_2(s_1,s_2)=H(s_2)/(s_1+s_2)$$

2DLT

- **Adding together, we obtain:**

$$L_{2D}\{h(|t-\tau|)\}=H(s_1,s_2)=\frac{H(s_1)+H(s_2)}{s_1+s_2}$$

Symbolic Bifrequency Input-Output System Representation



Black box representation of circularly symmetric linear systems

The 2DLT of General LTV Systems

Consider a SISO LTV system, initially at rest, described by:

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t)}{dt^i} = \sum_{k=0}^{m} b_k(t)\frac{d^k x(t)}{dt^k}$$

An input *x(.)* is applied to the system at time $\xi = t - \tau$

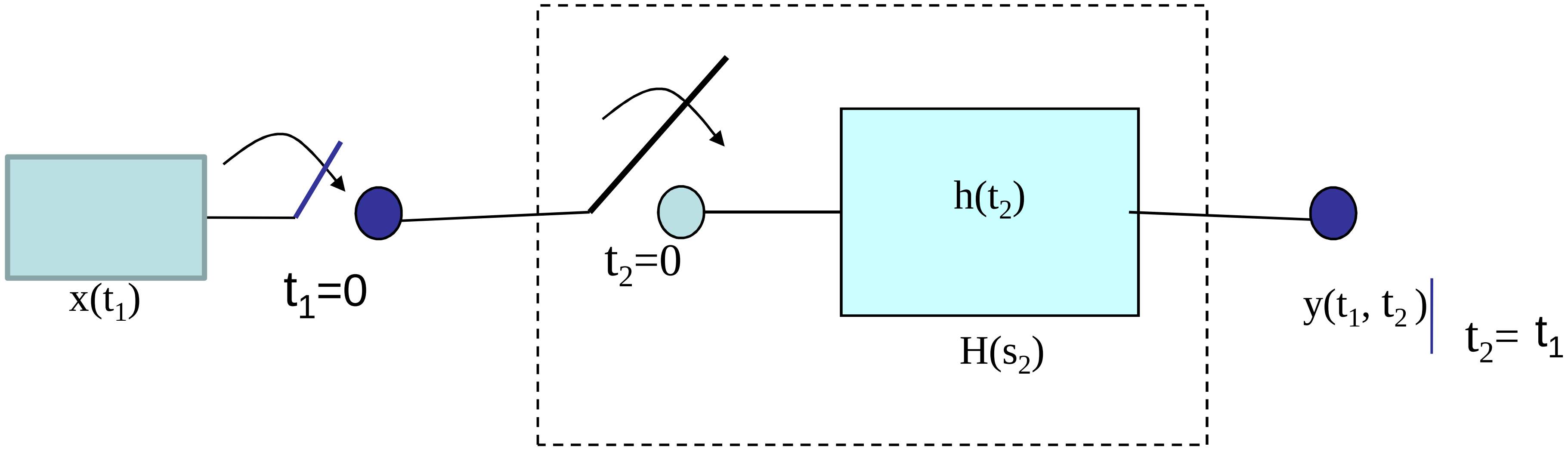
$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t,\tau)}{dt^i} = \sum_{k=0}^{m} b_k(t)\frac{d^k x(t-\tau)}{dt^k}$$

This may demand more initial conditions that the problem requires!

Taking a 2DLT, we obtain:

$$\sum_{i=0}^{n}\int_0^{\infty}\int_0^{\infty} a_i(t)\frac{d^i y(t,\tau)}{dt^i}e^{-s_2 t}e^{-s_1\tau}dt d\tau = \sum_{k=0}^{m}\int_0^{\infty}\int_{\xi=\tau_-}^{t} b_k(t)\frac{d^k x(t-\tau)}{dt^k}e^{-s_2 t}e^{-s_1\tau}dt d\tau$$

Synchronized System Representation



t_1 is the observation time of signal and t_2 is the application time to the system.

The 2DLT of Synchronized LTV Systems

- Consider a SISO LTV system described by:

$$\sum_{i=0}^{n} a_i(\tau)\frac{d^i y(t)}{dt^i} = \sum_{k=0}^{m} b_k(\tau)\frac{d^k x(t)}{dt^k}$$

- Taking Laplace transform with respect to τ, *we obtain:*

$$\sum_{i=0}^{n}\int_{-\infty}^{+\infty} a_i(\tau)e^{-s_2\tau}\frac{d^i y(t)}{dt^i}d\tau = \sum_{k=0}^{m}\int_{-\infty}^{+\infty} b_k(\tau)e^{-s_2\tau}\frac{d^k x(t)}{dt^k}d\tau$$

$$\sum_{i=0}^{n} A_i(s_2)\frac{d^i y(t)}{dt^i} = \sum_{k=0}^{m} B_k(s_2)\frac{d^k x(t)}{dt^k}$$

The 2DLT of Synchronized LTV Systems (Cont.)

- Taking a second transform with respect to *t*:

$$\sum_{i=0}^{n} A_i(s_2)\int_{-\infty}^{+\infty} \frac{d^i y(t)}{dt^i} e^{-s_1 t} dt = \sum_{k=0}^{m} B_k(s_2)\frac{d^k x(t)}{dt^k} e^{-s_1 t} dt$$

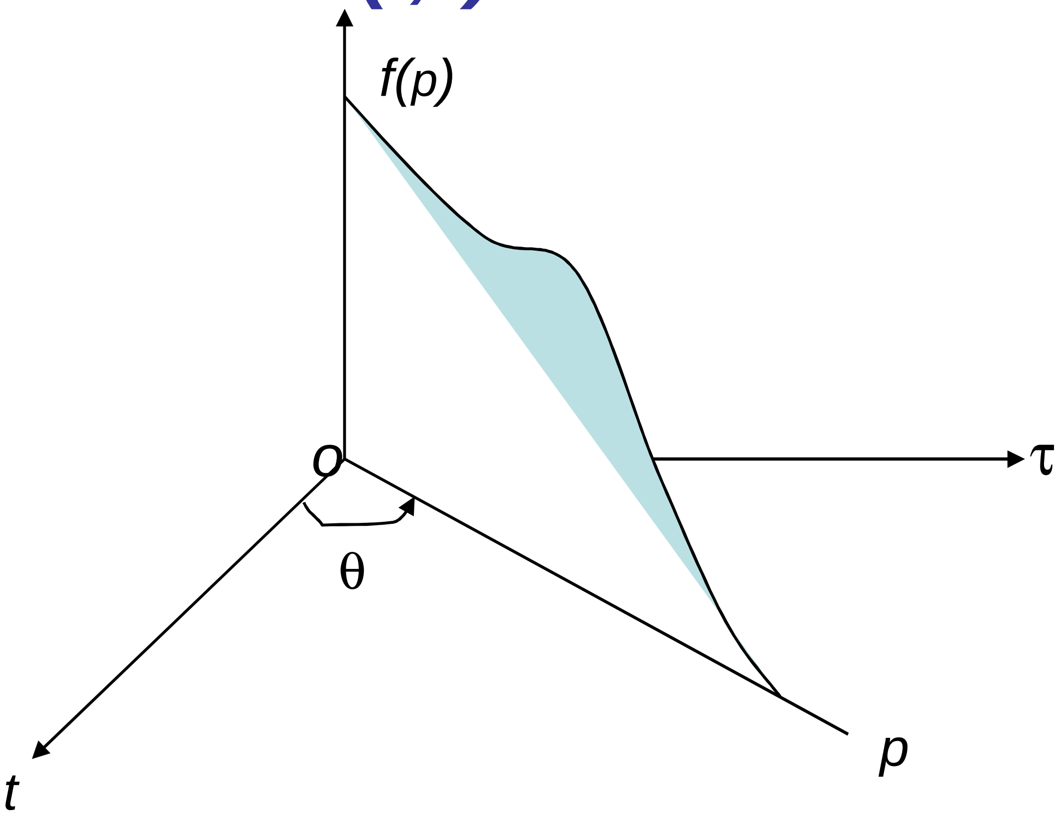
- If the system is initially at rest then $\frac{d^i y(t)}{dt^i} = 0$ for *i =0, 1, 2, . . . , n-1*.

$$\sum_{i=0}^{n} A_i(s_2) s_1^i Y(s_1) = \sum_{k=0}^{m} B_k(s_2)\left[s_1^k X(s_1) - s_1^{k-1} x(0) - s_1^{k-2} x^{(1)}(0) - \ldots - x^{(k)}(0) \right] =$$

$$= \sum_{k=0}^{m} B_k(s_2)\left[s_1^k X(s_1) - \sum_{j=0}^{k-1} s_1^{k-j} x^{(j)}(0) \right]$$

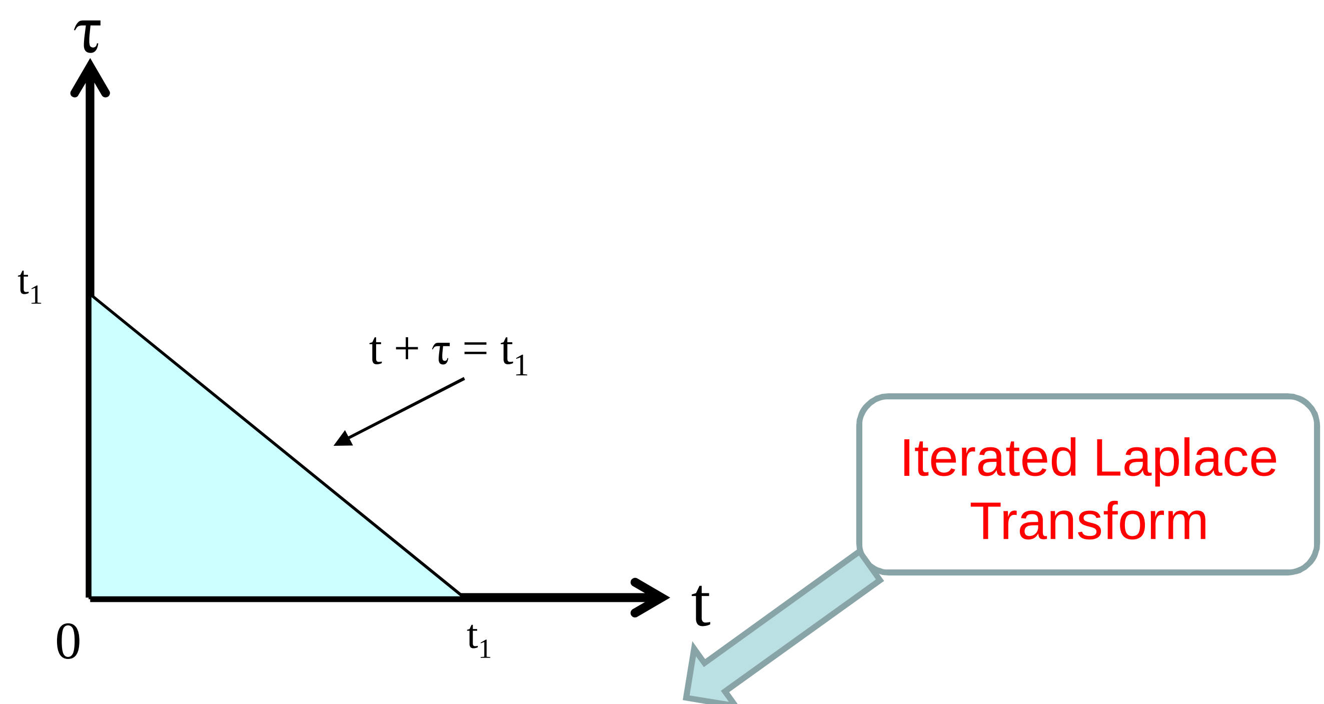
- Thus $$Y(s_1) = \sum_{k=0}^{m} B_k(s_2)\left[s_1^k X(s_1) - \sum_{j=0}^{k-1} s_1^{k-j} x^{(j)}(0) \right] \Bigg/ \sum_{i=0}^{n} A_i(s_2) s_1^i$$

Real-Variable Function Representation in (*t,τ*)-Plane



- A system function, which has the simple rotational property of circular symmetry is shown in this figure.
- Figure shows conversion of a one-dimensional profile of a system function of *p* to a two-dimensional function of a complex variable of *t* and τ.
- The common views of the independent signal and system functions are the projection over *t-axis* (pure real) and *τ-axis* (pure imaginary), respectively.

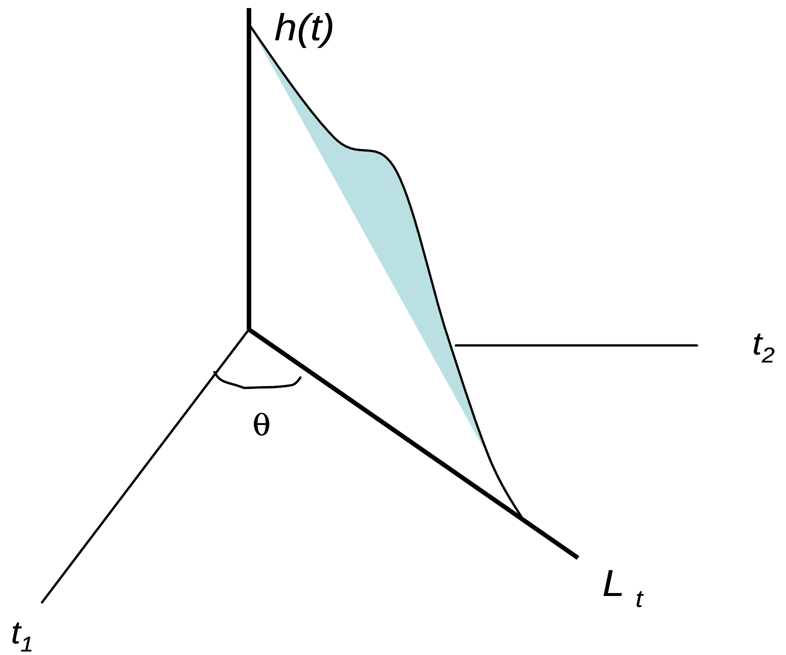
Circularly Symmetric System Transformation



$$Y(s) = \iint_{\Delta} h(t,\tau)x(\tau)e^{-s(t+\tau)}dtd\tau$$

Region over which the impulse-response *h(t, τ)* of a nonanticipative system is defined is the shaded area.

Circularly Symmetric Functions (1)



$$h(t,\theta) = h(t\cos\theta, t\sin\theta) \equiv h_{sym}(t) = \frac{1}{2}[h(t_1,t_2) + h(t_2,t_1)]$$

$$H(s,\phi) \equiv H_{sym}(s) = \frac{1}{2}[H(s_1,s_2) + H(s_2,s_1)]$$

Rotational property of a circularly symmetric function

Other Observations on LTV Systems

- **Mathematically speaking, *t* and *τ* represent time variables of an applied signal and the corresponding system.**
- **Variations of an input signal and an autonomous system are independent of each other.**
- **For circularly symmetrical systems, with no loss of generality, we can rewrite the response as:**

$$y(t,\tau) = e^{-\ln h(t,\tau)} x(t)$$

Generalized-Delay System Representation

- **If *h(t)* is a (piecewise) continuous function and bounded by a finite number, its 1st -order and higher-order derivatives exists.**

- **The system response can equivalently be written as:**

$$y(t) = e^{-\int_{\tau}^{t} \frac{h'(\xi)d\xi}{h(\xi)}} x(t)$$

- **The system response can be written more compactly as a generalized-delay operator:**

$$y(t) = e^{-g(t,\tau)} x(t)$$

The Hankel Transform

- The Hankel transform is compatible with LTV systems described by a general **Bessel** equation given as:

$$\left[\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\left(\frac{n^2}{t^2}\right)\pm a^2\right]^N y(t)=x(t)$$

- The Hankel transform pairs are symmetric because it deals with symmetric functions.
- The 2DLT of a circularly symmetric function with the property $\lim_{t\to\infty} h(t) \to 0$ that is a Hankel transform of order zero.
- This property is quite useful in application of Hankel transforms to LTV systems.

The Mellin Transform

- The Mellin transform is compatible with LTV systems characterized by a general **Euler-Cauchy** equation given as:

$$\sum_{i=0}^{n} a_i t^i \frac{d^i y(t)}{dt^i} = x(t)$$

- The impulse response of this nonanticipative Euler-Cauchy LTV system is:

$$h(t,\tau) = \frac{1}{t} g\left(\frac{t}{\tau}\right) u(t-\tau)$$

  - where *g(.)* is the impulse response of a prototype LTI system obtained by changing the time scale *t* → *- ln t*

- The 2DLT in this case becomes the following Mellin Transform pairs:

$$M\{h(t)\} = \int_0^{+\infty} h(t) t^{s-1} dt \qquad M^{-1}\{H(s)\} = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} H(s) t^{-s} ds$$

Conclusions

- **The 2DLT techniques are applicable to LTV systems.**
- **This investigation justifies application of 2DLT as an operational calculus for system characterization, especially for analog signal processing problems.**
- **This approach allows, in effect, two-dimensional transform techniques to be used for the time-varying systems in the same manner that the conventional frequency-domain techniques are used in connection with fixed systems.**
- **The 2DLT, Mellin transform, and Hankel transform can be derived from the two-dimensional Fourier transform.**
- **The work presented here opens several areas for further investigations in theory of variable systems.**

For Further Information

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Questions?

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Dr. Erfani has published more than 70 technical papers, holds three patents, and is the Senior Technical Editor of the *Journal of Network and Systems Management* and an associate editor for *Computers & Electrical Engineering: An International Journal*.

He received a combined B.Sc. and M.Sc. degree in Electrical Engineering from the University of Tehran in 1971, and M.Sc. and Ph.D. degrees, also in Electrical Engineering, from Southern Methodist University in 1974 and 1976, respectively. He was a Member of Technical Staff at Bell Labs of Lucent Technologies in Holmdel, New Jersey, from 1985 to 2001.