Elliptic Curve Cryptography

Ashkan Hosseinzadeh Namin

Research Centre for Integrated Microsystems

Electrical and Computer Engineering

UNIVERSITY OF WINDSOR

Outline

- **Available Public key Cryptographic Techniques**
  - Motivation
  - Introduction
  - Symmetric Key and Asymmetric key Cryptography
  - RSA Cryptosystem
  - El Gamal Cryptosystem
- **Elliptic Curve Cryptography**
  - Elliptic Curve Definition
  - Point Addition and scalar multiplication
  - Elliptic Curve cryptosystem
  - Elliptic Curve Discrete Logarithm Problem
  - Comparison between Public key Cryptosystems
- **Conclusions**

Motivation

- **1976 Diffie and Hellman, first idea**
- **Efficient way to achieve secure data exchange between two unfamiliar parties**
- **RSA, El Gamal, ECC Cryptosystems**
- **With the same security level, smaller key size for ECC**
- **ECC implementations require less power, less memory**
- **Attractive for constrained devices like wireless devices and smart cards and handheld computers.**

Introduction

- **Cryptography : Greek word means Secret writing**

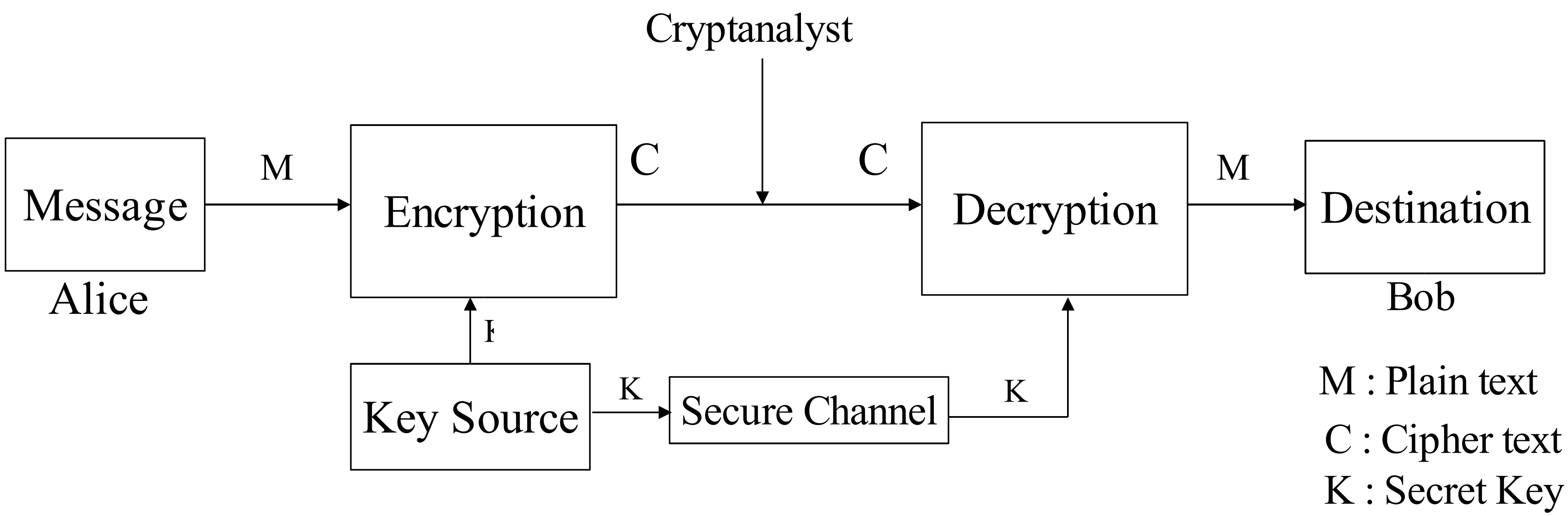
  Scrambling of data so that only someone with the necessary **key** can unscramble it. Used for secure data transmission and storage.

- **Cryptanalysis** : Deals with the breaking of an encrypted data (scrambled data) to recover information

- **Two main categories of cryptography**
  - Symmetric Cryptography or Secret Key
  - Asymmetric Cryptography or Public Key

Symmetric Key Cryptography

- **Alice and Bob agree on encryption method and a key**
- **Alice encrypts the message with the key and sends it to Bob**
- **Bob uses the same key to decrypt the message**



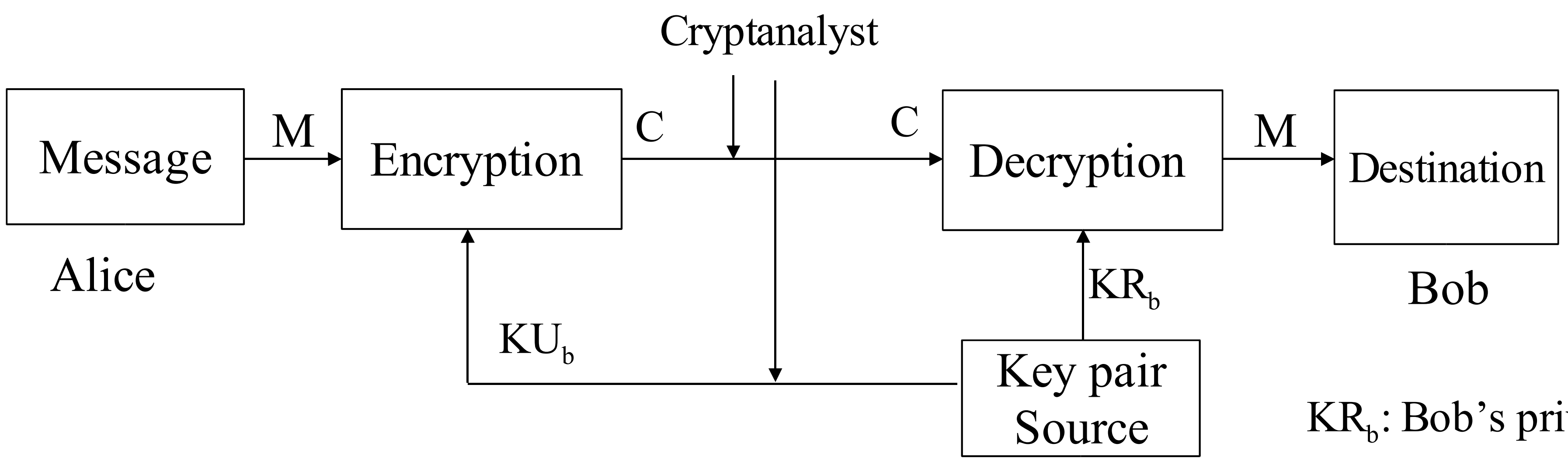
Symmetric Key Cryptography

- **Advantages**
  - High speed and high throughput
  - Short key size (> 128 bits)
  - Extensive history

- **Disadvantages**
  - The key must be remain secret at both ends
  - In a large network there are many key pairs to be managed (For n nodes $\frac{n \times (n-1)}{2}$ keys are required).

Asymmetric Key Cryptography

- **Bob generates a key which makes public.**
- **Bob uses his public key to determine a second key which is his private key and keeps it secret.**
- **Alice uses Bob's public key to encrypt a message for him.**
- **Bob uses his private key to decrypt this message**



KR_b: Bob's private key

KU_b: Bob's public key

Asymmetric Key Cryptography

- **Advantages**
  - Only the private key must be kept secret
  - Easier key administration on a network
  - In a large network, number of keys is smaller than symmetric-key scenario (n pairs of keys).

- **Disadvantages**
  - Small throughput
  - Larger Key size than symmetric-key encryption (160-1024 bits)
  - Public key schemes have their security based on some hard mathematical problems
  - Does not have as extensive a history (Started in 1970's)

Asymmetric Cryptography Techniques

- **The concept was introduced in 1976 by Diffie and Hellman as an algorithm for key exchange (they didn't come up with a practical cryptographic system)**

- **Today three different cryptographic systems are considered both secure and efficient.**
  - RSA (Based on integer factorization system)
  - El Gamal (based on discrete logarithm system)
  - Elliptic Curve (based on elliptic curve discrete logarithm problem)

- **All these cryptographic systems rely on the difficulty of a mathematical hard problem for their security and modular arithmetic plays a central role in their implementations.**

Asymmetric Cryptography Timeline

- **In 1978, L.M Adleman, R.L. Rivest and A. Shamir propose the RSA encryption method as the first public key algorithm. This algorithm is currently the most widely used.**

- **In 1985, Taher El Gamal proposed the discrete logarithm problem. In 1991 Schnorr discovered a variant Gamal's work which offers more efficiency. U.S government Digital Signature Algorithm is based on this technique.**

- **In 1985, Neil Koblitz and Victor Miller independently proposed the Elliptic Curve Cryptosystem (ECC). ECC is the strongest public key cryptographic system known today.**

RSA

- **Bob chooses two primes p and q and calculates $n=p\times q$**
- **Bob chooses e with $gcd(e,(p-1)\times(q-1))=1$**
- **Bob calculates d with $d\times e=1 \pmod{(p-1)\times(q-1)}$**
- **Bob makes n and e public, and keeps p,q,d secret**

- **Alice encrypts m as $c=m^{e} \pmod n$**
- **Bob decrypts by calculating $m=c^{d} \pmod n$**

- $m = c^{d} = m^{(d\times e)} = m^{(1)} = m \qquad \bmod n$

RSA

- **RSA security relies on the difficulty of the Integer Factorization problem**

- **Integer Factorization problem :**

  given a large prime number $n=p\times q$ factor n into it's prime numbers

- RSA efficiency rests on the speed of performing exponentiation modulo n.

- Up to 2003 the largest RSA modulus factored is a 530 bit binary number.

El Gamal

- **Bob chooses prime p and a primitive root α and makes them public**
- **Bob also chooses a secret integer A and calculates $B=(\alpha)^{A} \mod p$**
- **Bob public key is (p, α ,B) and his private key is A**
- **Alice chooses a random integer k and calculates $K=(\alpha)^{k}$**

- **Alice encrypts m as $C_1= \alpha^{K}, C_2=B^{K} \times m \mod p$**
- **Bob decrypts by calculating $C_2 \times (C_1)^{-A}$**

- **$m = C_2 \times (C_1)^{-A} = B^{K} \times m \times (\alpha^{K})^{-A} = (\alpha^{A})^{K} \times m \times (\alpha^{K})^{-A} = m \mod p$**

El Gamal

- **El Gamal security relies on the difficulty of the Discrete Logarithm problem.**

- **Discrete Logarithm problem :**

  Given pair g and y and prime number p such that $y = g^x \pmod{p}$ determine integer x

- El Gamal efficiency rests on the speed of performing modular exponentiation modulo p.

- Up to 2003 the largest DLP solved is a 397 bit binary number.

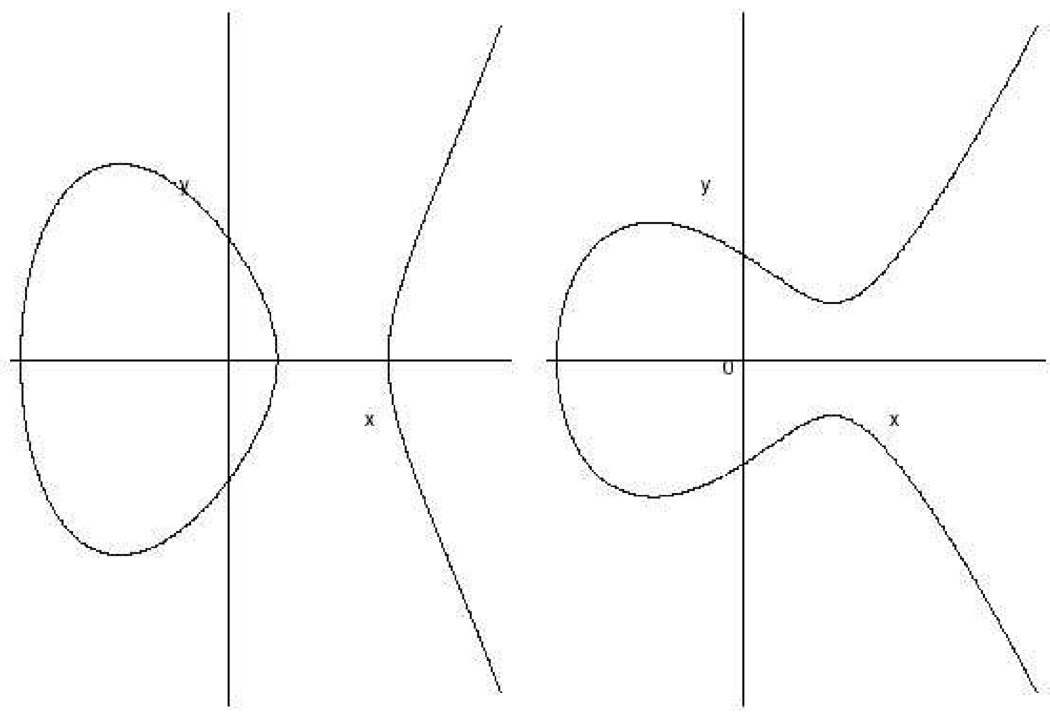
Elliptic Curve Definition

- An Elliptic Curve is the graph of equation of the form

$$y^2 = x^3 + ax + b$$

(we assume that the curve has no multiple roots $4a^3+27b^2\neq0$)

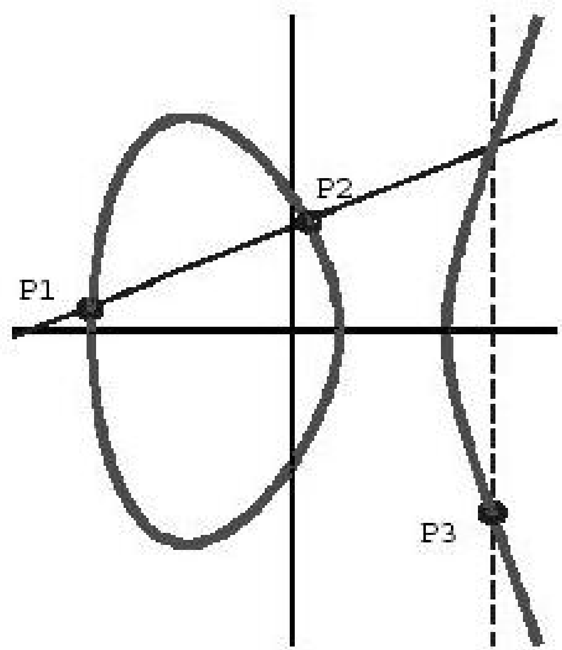
When we are working with real Numbers graph E has one of the two possible forms (it can have one real root or three real roots).



Elliptic Curve Point Addition

- if (x,y) satisfy the elliptic curve equation then p=(x,y) is a point on the elliptic curve
- Suppose P_1 and P_2 are both points on the elliptic curve then
  $P_1 + P_2$ is always another point on the elliptic curve which is defined as

Draw a line through P_1 and P_2 (if $P_1 = P_2$ take the Tangent line). The line intersects the curve in a third Point. Reflect that point through the x-axis to find $P_3 = P_1 + P_2$



Elliptic Curve Point Addition

- **For curve $y^2 = x^3 + ax + b$**

- **Point Addition $P(x_1,y_1) \neq Q(x_2,y_2)$**

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2 \qquad y_3 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) \times (x_1 - x_3) - y_1$$

- **Point Doubling $P(x_1,y_1)$**

$$x_3 = \left(\frac{3x_1^2 + a}{2y_1}\right)^2 - 2x_1 \qquad y_3 = \left(\frac{3x_1^2 + a}{2y_1}\right) \times (x_1 - x_3) - y_1$$

Elliptic Curve Scalar Multiplication

- **Scalar multiplication is the dominant computation part of ECC**
- **It computes k×P for a given point P and integer k.**
- **Q = k×P = (P + P + … + P) ((k-1) addition)**

- **There are different methods for speeding up this process, The most common one is the Binary Method (also called Double and Add Method)**

For i = 0 to n-1
If b_i=1 then Q = Q + P
P = P + P
End

$$K = \sum_{i=0}^{n-1} b_i * 2^i$$

$$b_i = 0,1$$

Elliptic Curve & Finite Field

- Elliptic curve calculations are usually defined over finite field

The finite field is prime field GF(P)

The elements are {0,1,2,…,p-1}

all operations are modulo p

The finite field is a binary polynomial field $GF(2^m)$

The elements are binary polynomials

all operations are modulo 2

$$x = a_{m-1}X^{m-1} + a_{m-2}X^{m-2} + \ldots + a_1X + a_0 \qquad a_i = \{0,1\}$$

Defining the curve over Binary Field will speed up the calculations

Elliptic Curve Cryptosystem

- **Bob chooses the curve E and pint P on the curve**
- **Bob chooses integer d and calculates $Q=d\times P$ and makes it public**
- **Alice maps the plaintext m to point M on curve**
- **Alice chooses a random integer k**

- **Alice encrypts M as $C_1=k\times P$, $C_2= M + k\times Q$**
- **Bob decrypts by calculating $M=C_2 - d\times k\times P$**

- $\mathbf{M = C_2 - d\times k\times P = M + k\times Q - d\times k\times P = M + k\times Q - d\times Q = M}$

(Elliptic curves, points on them and mapping formats are standardized)

Elliptic Curve Discrete Logarithm Problem

- **Elliptic Curve security relies on the difficulty of the Elliptic Curve Discrete Logarithm problem (ECDLP)**

- **Elliptic Curve Discrete Logarithm problem:**
- ECDLP is the inversion to scalar multiplication and defined as
  Given points Q and P, find the integer k such that $Q = K \times P$

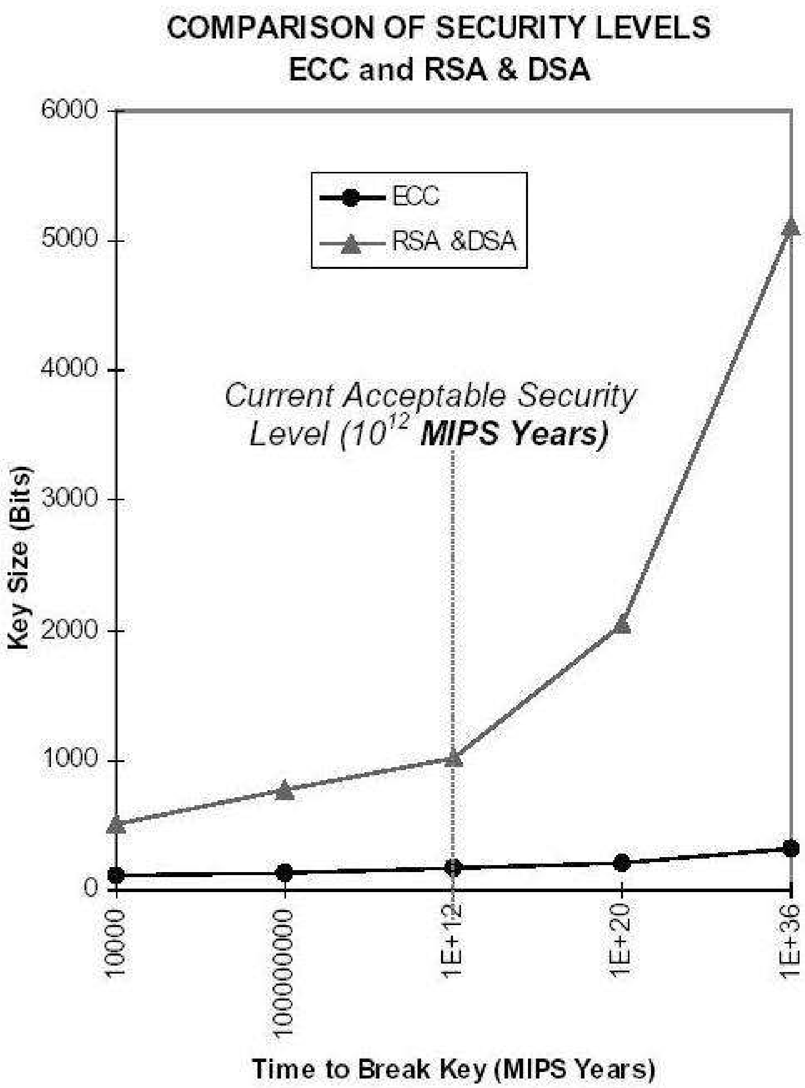
- ECC efficiency rests on the speed of calculating $k \times P$ for some integer k and a point P on the curve.

Up to 2003 the largest ECDLP solved is a 109 bit prime field binary number.

Comparison between Public Key Cryptosystems

Secure system : It is generally accepted that 10^{12} MIPS years represents reasonable security at this time.

MIPS year : computing time of one year on a machine capable of performing one million instructions per second.



Comparison between Public Key Cryptosystems

- To achieve reasonable security **today**, RSA and DSA (El Gamal) should employ a 1024 modulus, while a 160 bit modulus should be sufficient for ECC.
- The security gap between the systems grows as the key size grows. For example a 300 bit ECC provides the same security as a 2000 bit RSA or DSA
- Shorter keys reduce storage space for keys and faster computation speed which makes ECC suitable for constrained applications where computational power and bandwidth is limited.

Conclusions

- Information security through public key cryptography is required for electronic transactions for unfamiliar parties
- Three different approaches are RSA, El Gamal and ECC
- ECC offers the highest security (strength per bit)
- Security gap between systems grows as the key size grows
- ECC is suitable for constrained applications such as smart cards, tokens, wireless communication devices