# Computer Aided 2-D Recursive filter stability analysis and stabilization

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# Introduction of filter stability

- **1.Definition:**
- A stable filter in which the convolution of the impulse response with some bounded input sequence will always yield a bounded output.
- **2.Motivation:**
- Stability has the first priority in the filter design.
- It is very difficult to test high dimensional filters' stability.
- There is still no perfect solution to the 2-D filter's stabilization problem.

# 1-D recursive filter's stability test

- 1. Shur-Cohn Stability Criterion.
- 2. Schur-Colhn-Fujiwara Stability Criterion.
- 3. Jury-Marden Stability Criterion.

# Jury-Marden Stability Criterion

Consider a filter characterized by the transfer function:

$$H(z) = \frac{N(z)}{D(z)} \qquad N(z) = \sum_{i=0}^{N} a_i z^{M-i} \qquad D(z) = \sum_{i=0}^{M} b_i z^{N-i}$$

| Row | Coefficients | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | $b_0^0$ | $b_1^0$ | $b_2^0$ | $\cdots$ | $b_{M-2}^0$ | $b_{M-1}^0$ | $b_M^0$ |
| 2 | $b_M^0$ | $b_{M-1}^0$ | $b_{M-2}^0$ | $\cdots$ | $b_2^0$ | $b_1^0$ | $b_0^0$ |
| 3 | $b_0^1$ | $b_1^1$ | $b_2^1$ | $\cdots$ | $b_{M-2}^1$ | $b_{M-1}^1$ | |
| 4 | $b_{M-1}^1$ | $b_{M-2}^1$ | $b_{M-3}^1$ | $\cdots$ | $b_1^1$ | $b_0^1$ | |
| 5 | $b_0^2$ | $b_1^2$ | $b_2^2$ | $\cdots$ | $b_{M-2}^2$ | | |
| 6 | $b_{M-2}^2$ | $b_{M-3}^2$ | $b_{M-4}^2$ | $\cdots$ | $b_0^2$ | | |
| $\vdots$ | $\vdots$ | | | | | | |
| $2M-3$ | $b_0^{M-2}$ | $b_1^{M-2}$ | $b_2^{M-2}$ | | | | |

$$b_i^k = \begin{bmatrix} b^{k-1}{}_0 & b^{k-1}{}_{N-i} \\ b^{k-1}{}_N & b^{k-1}{}_i \end{bmatrix}$$

$(1) D(1) > 0$

$(2) (-1)^N D(-1) > 0$

$(3) b_0{}^0 > \left| b_N{}^0 \right|$

$\left| b_0{}^i \right| > \left| b_{N-i}{}^i \right|$ When i>0 and i<=2M-3

# Stabilization of one-dimensional recursive digital filters

- For an unstable recursive filter:

$$H(z) = \frac{\sum_{i=0}^{M} a_i z^i}{\prod_{i=1}^{K} (z - r_i e^{j\theta}) \prod_{i=K+1}^{M} (z - \frac{1}{r_i} e^{j\theta})} \qquad |r_i| < 1$$

- An all pass filter can stabilize the system:

$$H(z) = \prod_{i=K+1}^{M} \frac{z - \frac{1}{r_i} e^{i\theta_i}}{z - r_i e^{i\theta_i}}$$

# Stability and stabilization of 2-D recursive filters

- 1)Fundamental theorem of algebra is not applicable to two-variable functions, which means denominator factorization is not always possible.
- 2)Stabilization is impossible for the high dimensional systems.
- 3) For 2-D filters even the numerator of the filter's transfer function can effect the stability of the filter.

# Stability Theorem about 2-D filter (1)

- Theorem 1: Given that $D(z_1, z_2)$ is a polynomial in z1 and z2, a necessary and sufficient condition for the coefficients of the expansion of
- $H(z_1, z_2) = 1 / D(z_1, z_2)$ in negative power of z1 and z2 to converge absolutely, and hence for h(m,n) to be absolutely summable, is
- $D(z_1, z_2) \neq 0 \ for \ \bigcap |z_i| \geq 1$
- *The test is done by assigning values to the variable $z_1$ and finding the roots of $D(z_1,z_2)=0$ as a function of $z_2$. The testing for stability as stated above is very tedious to apply, since it involves mapping an infinite number of points from the $z_1$*

# Stability Theorem about 2-D filter (2)

- Huang's two conditions:
- A causal 2-D recursive filter with a z-transfer function

$$H(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1, z_2)}$$

- Is stable if and only if:

$$D(z_1, 0) \neq 0, |z_1| \geq 1$$ Criterion 1

$$D(z_1, z_2) \neq 0, |z_1| = 1 \cap |z_2| \geq 1$$ Criterion 2

# Modified Jury Stability Test Method---For Criterion 1

THEOREM 3.6. *Let $f(z)$ be the nth degree polynomial given by*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{3.74}$$

*where coefficients $a_i$, $i = 0, 1, \ldots, n$ are complex numbers. The roots of $f(z)$ are inside the unit circle if and only if*

$$b_0 < 0, \qquad c_0 > 0, \qquad d_0 > 0, \ldots, g_0 > 0, \ldots, t_0 > 0 \tag{3.75}$$

*where $b_0, c_0, \ldots, t_0$ are obtained from the modified Jury's table formed as follows:*

| | | | | | | |
|---|---|---|---|---|---|---|
| $z^0$ | $z^1$ | $z^2$ | $\cdots$ | $z^{n-2}$ | $z^{n-1}$ | $z^n$ |
| $a_0$ | $a_1$ | $a_2$ | $\cdots$ | $a_{n-2}$ | $a_{n-1}$ | $a_n$ |
| $a_n^*$ | $a_{n-1}^*$ | $a_{n-2}^*$ | $\cdots$ | $a_2^*$ | $a_1^*$ | $a_0^*$ |
| $b_0$ | $b_1$ | $b_2$ | | $b_{n-2}$ | $b_{n-1}$ | |
| $b_{n-1}^*$ | $b_{n-2}^*$ | $b_{n-3}^*$ | | $b_1^*$ | $b_0^*$ | |
| $c_0$ | $c_1$ | $c_2$ | | $c_{n-2}$ | | |
| $c_{n-2}^*$ | $c_{n-3}^*$ | | | $c_0^*$ | | |
| $\vdots$ | | | | | | |
| $r_0$ | $r_1$ | | | | | |
| $r_1^*$ | $r_0^*$ | | | | | |
| $t_0$ | | | | | | |

(3.76)

*where*

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n^* & a_k^* \end{vmatrix} \qquad \text{and} \qquad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1}^* & b_k^* \end{vmatrix}$$

*and $a_k^*$ is the complex conjugate of $a_k$.*

# Modified Jury Stability Criterion---For Condition 2

We can reconstruct a 2-d function $D(z_1, z_2) = \sum_{i,j=0}^{N} b_{ij} z_1^{\,i} z_2^{\,j}$ as below:

$$D(z_1, z_2) = \sum_{i=0}^{N} a_i(z_1) z_2^{\,i} \qquad a_i(z_1) = \sum_{i=0}^{M} b_{ij} z_1^{\,i}$$

1. All of the $b_{ij}$ are real, and then $a_j^*(z_1) = a_j(z_1^*)$
2. $Z_1$ is restricted to the boundary $|z|=1$, and thus $|z_1 z_1^*|^l = 1$ for all l.
   Then all of the judging functions ($b_j, c_j, d_j, r_j$) can be expressed as function of (z1*+z1)

$$(z_1^* + z_1) = 2X$$

$$(z_1^{*2} + z_1^{2}) = 4X^2 - 2$$

$$(z_1^{*3} + z_1^{3}) = 8X^3 - 16X$$

3. Since -1<=X<=1 , it is easy for us to judge whether:

$$b_0(X) < 0 \qquad c_0(X) > 0 \qquad r_0(X) > 0$$

# Conventional stabilization of 2D recursive filter

- NO PERFECT SOLUTION.

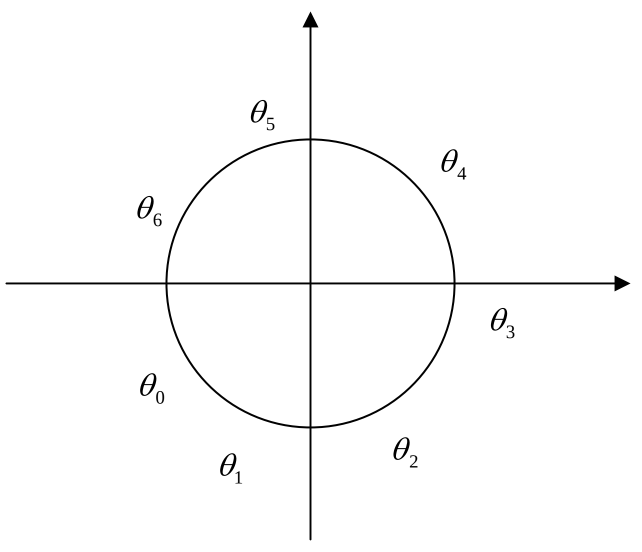
# Proposed computer aided algorithm

- Computer aided analysis of Huang's theorem 2

$$D(z_1, z_2) \neq 0, |z_1| = 1 \cap |z_2| \geq 1$$

$$z_1 = e^{j\theta_i} \qquad \theta_i \in [-\pi, \pi] \qquad i \in [1, N]$$

N is the total number of the sample angles



# Computer aided analysis of Huang's criterion 2

For the denominator function $D(z_1, z_2) = 0$

- Since $|z_1| = 1$, we substitute $z_1 = e^{j\theta_i}$

$$D(z_2, \theta_i) = 0 \qquad D(z_1, z_2) = \sum_{j=0}^{N} a_j(\theta_i) z_2{}^{i}$$

Here $\theta_i$ is a constant to the function $z_2$

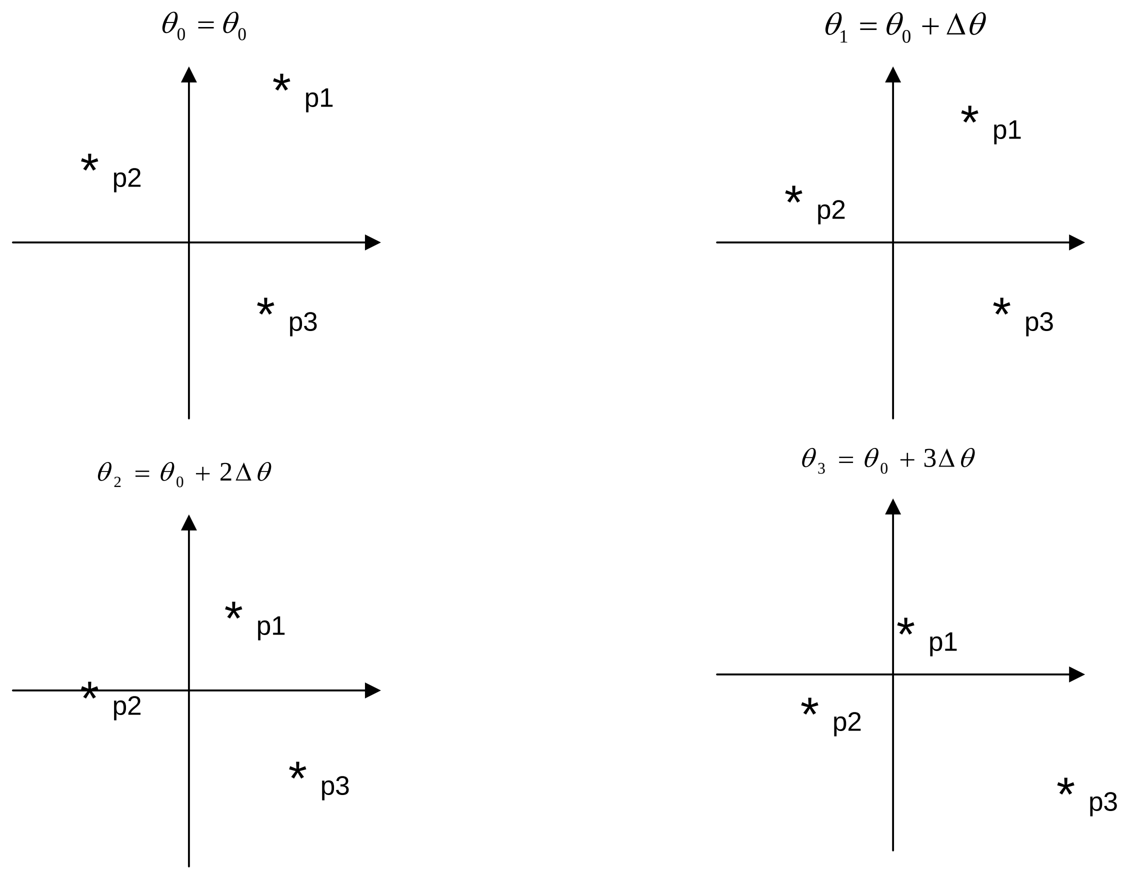
With the aid of computer, it's easy for us to factorize above function as below :

$$\prod_{j=1}^{N} (z_2 - p_j(\theta_i)) = 0$$

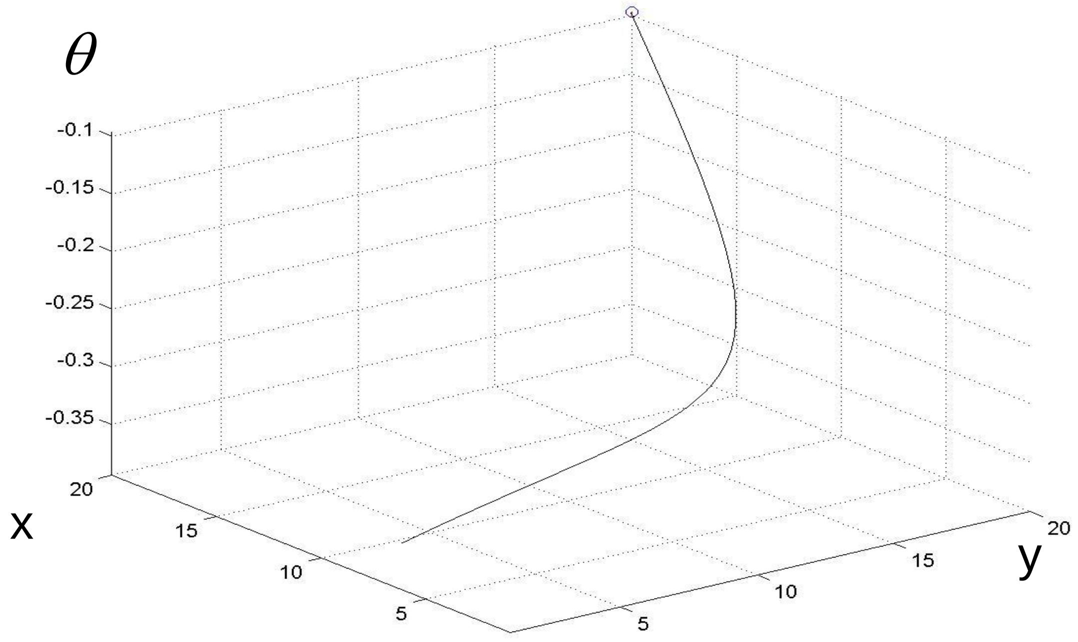
N is the order of the polynomial of $Z_2$

# Tracking the poles with the aid of computer

Here $\Delta\theta$ is a small value. So when we change the angle from 0 to pi, Then the N complex poles will move from the start to end.



# A demonstration of the complex value $Z_2$ changing with the $\theta$



# Feasibility of tracking the poles(1)

For $z_{2j} = p_j(\theta_i)$ : $\theta \in (0, \pi)$

1. $Z_2$ is finite. Since: $D(z_1, z_2) = \sum_{i=0}^{N} a_i(z_1) z_2{}^i = 0$ while $|Z_1| = 1$ obviously $a_i(z_1)$<infinite

2. For any $\theta \in (0, \pi)$, there is a an available complex value Z

3. The $p_j(\theta_i)$ is continuous function. For any $\theta \in (0, \pi)$ $p_j(\theta_i^+) = p_j(\theta_i^-)$

# Feasibility of tracking the poles(2)

- If $D(z_1, z_2) = (z_1 - p_1)^k D'(z_1, z_2) = 0$ $|p_1| = 1$ Then any $z_2$ can be solution to the equation.
- if $D(z_1, z_2) = (z_1 - p_1)^k D'(z_1, z_2) + c = 0$ $|p_1| = 1$ while c is constant and isn't zero. There is no solution to $z_2$.

It is always possible to decide whether there are factors of

$(z_1 - p_1)$ or $(z_2 - p_2)$ in the function $D(z_1, z_2)$

# Feasibility of tracking the poles(3)

If $D(z_{1i}, z_{2i}) = 0 \qquad z_{1(i+1)} = z_{1(i)} + \Delta z_1 \qquad z_{2(i+1)} = z_{2(i)} + \Delta z_2$

$$D(z_{1(i+1)}, z_{2(i+1)}) = 0$$

$$D(z_{1(i+1)}, z_{2(i+1)}) = D(z_{1i}, z_{2i}) + \frac{\partial D(z_1, z_2)}{\partial z_1} \times \Delta z_1 + \frac{\partial D(z_1, z_2)}{\partial z_1} \times \Delta z_1$$

$$= \frac{\partial D(z_1, z_2)}{\partial z_1} \times \Delta z_1 + \frac{\partial D(z_1, z_2)}{\partial z_2} \times \Delta z_2$$

$$\frac{\partial D(z_1, z_2)}{\partial z_1} \times \Delta z_1 = -\frac{\partial D(z_1, z_2)}{\partial z_2} \times \Delta z_2$$

$$\Delta z_2 = -\Delta z_1 \times \left(\frac{\partial D(z_1, z_2)}{\partial z_1}\right) / \left(\frac{\partial D(z_1, z_2)}{\partial z_2}\right)$$

# Feasibility of tracking the poles(4)

$$\Delta z_2 = -\Delta z_1 \times \left(\frac{\partial D(z_1, z_2)}{\partial z_1}\right) / \left(\frac{\partial D(z_1, z_2)}{\partial z_2}\right) \qquad \Delta z_1 = \varepsilon$$

Case 1: $\frac{\partial D(z_1, z_2)}{\partial z_1} \neq 0$ $\quad \frac{\partial D(z_1, z_2)}{\partial z_2} \neq 0$ $\quad \Delta z_2 = \varepsilon$

Case 2: $\frac{\partial D(z_1, z_2)}{\partial z_1} \neq 0$ $\quad \frac{\partial D(z_1, z_2)}{\partial z_2} = 0$

$$D(z_1, z_2) = (z_1 - z_{1i}) \; D'(z_1, z_2) + c = 0$$

Case 3: $\frac{\partial D(z_1, z_2)}{\partial z_1} = 0$ $\quad \frac{\partial D(z_1, z_2)}{\partial z_2} \neq 0$ $\quad \Delta z_2 = 0$

Case 4: $\frac{\partial D(z_1, z_2)}{\partial z_1} = 0$ $\quad \frac{\partial D(z_1, z_2)}{\partial z_1} = 0$

$$D(z_1, z_2) = D_1(z_1, z_2)^k D_2(z_1, z_2)$$

$z_1$,$z_2$ are the zeros of the $D_1(z_1,z_2)$, this is easy to be identified by our method since the poles are overlapped.

# Approximation of the poles' functions

- We can get N functions about the magnitudes of the poles, the variable is the angle the z1 rotating around the unit circle.

$$\theta \in (0, \pi)$$

$$A_j = \left| z_{2j} \right| = \left| p_j(\theta) \right| = \sqrt{real(p_j(\theta)^2 + image(p_j(\theta)^2} = Function_j(\theta)$$

$j \in (1, N)$ N is the number of the poles in the $z_2$ function

We sample M values from 0 to pi, then we can get the M $A_{jm}$.

# Implementation of Taylor series(1)

- According to Taylor function we can approximate the function into polynomial function. We have n individual such functions.

$$A = Function(\theta) = \sum_{i=0}^{M} a_i \times \theta^i$$

For each function we have M sample $\theta_m$

$$\begin{bmatrix} \theta_1^{\,m} & \overline{\theta_1^{\,m-1}} & \cdot & 1 \\ \theta_2^{\,m} & \theta_2^{\,m-1} & \cdot & 1 \\ \theta_3^{\,m} & & & 1 \\ & & \cdot & 1 \\ \theta_{m+1}^{\,m} & \theta_{m+1}^{\,m-1} & \cdot & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \\ a_m \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \\ \\ A_{m+1} \end{bmatrix}$$

# Implementation of Taylor series(2)

- Obviously it's easy for us to get the $a_i$ in the below function

$$A = Function(\theta) = \sum_{i=0}^{M} a_i \times \theta^i \qquad \theta \in (0, \pi)$$

Then it is easy for us to get the range of A when the angle has limited area.
The minimum or maximum happen in below two cases:

1) When $\theta$ is at the end of the area.
2) When $\frac{\partial A}{\partial \theta} = 0$

# Procedure of the stability test

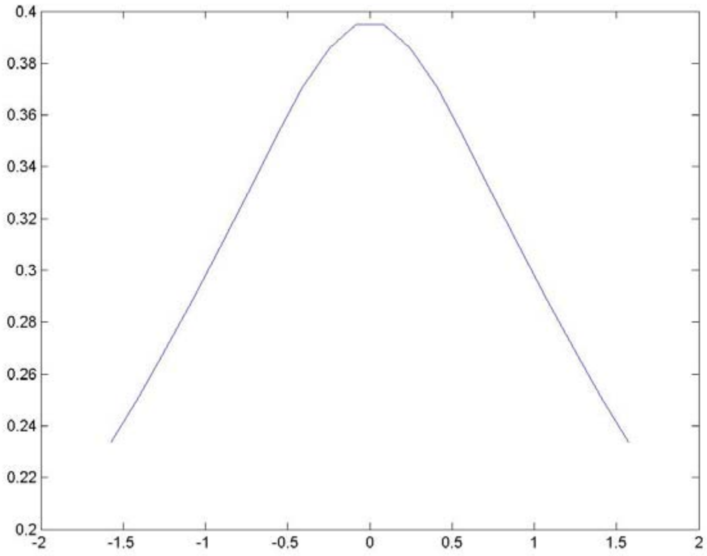
- 1) Convert a 2-D polynomial into a matrix

$$D(z_1, z_2) = \sum a_{ij} z_1{}^i z_2{}^j$$

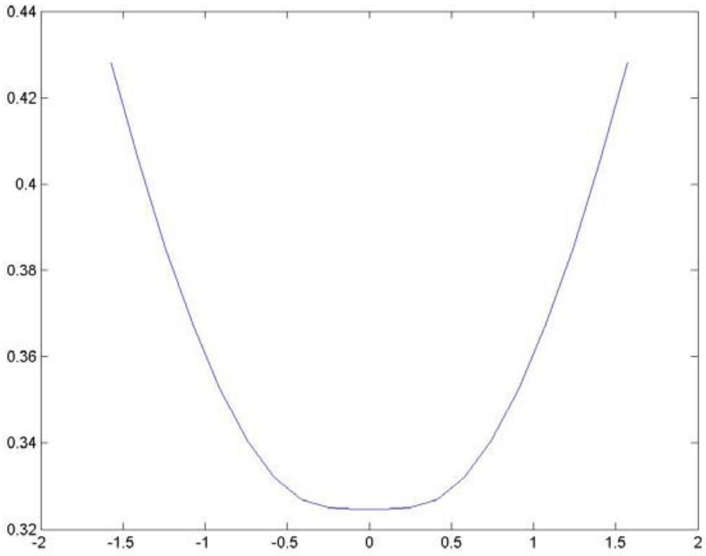
Then we can get a matrix $A = \lfloor a_{ij} \rfloor$

A=[-0.1 -0.01 -0.2;
-0.01 -0.24 -0.1;
0.02 0.3 1] ;

# The simulation result




Curve of the pole 1




Curve of the pole 2

# Stabilization of two-dimensional recursive digital filters(1)

- For an 2D recursive filter: $H(z_1, z_2) = \dfrac{N(z_1, z_2)}{D(z_1, z_2)}$
- Can be converted into function like:

$$H(z_1, z_2) = \frac{N(z_1, z_2)}{\prod_{i=0}^{n} (z_2 - p_i(z_1))}$$

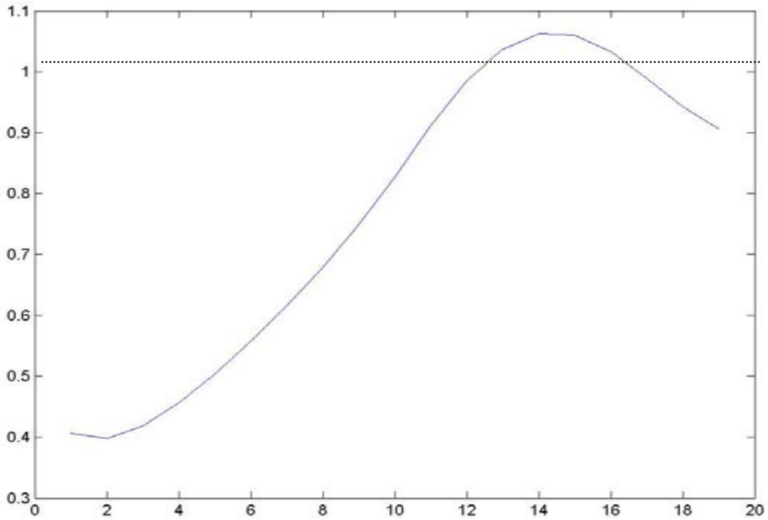
The functions of $p_i(z_1)$ can be use the sampling the $z_1$ around the unit circle ,with the same method as before we can get a Taylor function for each pole.

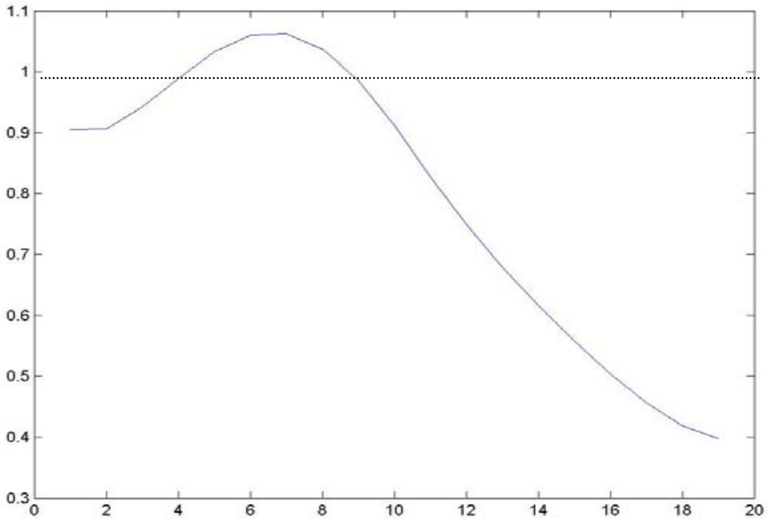
$$p_i(z_1) = \sum_{i=0}^{M} a_i \times z_1^{\ i}$$

# Stabilization of two-dimensional recursive digital filters(2)

- For different poles the corresponding unstable angles can be found and can be modify the coefficient to get the new response to guarantee every pole is stable.

# Illustration of modified amplitude response.



# Future job

- 1. When the number of sample is high, it's difficult to compute (at least for the matlab) to get the accurate inverse matrix.
- 2. I haven't find the error range of the Taylor series.
- 3. Do some more simulations about the stabilization.