Computer Aided 2-D Recursive filter stability analysis and stabilization

Songtao Huang
Supervisor: Dr. Ahmadi
Electrical and computer engineering
University of Windsor
Introduction of filter stability

1. Definition:
A stable filter in which the convolution of the impulse response with some bounded input sequence will always yield a bounded output.

2. Motivation:
- Stability has the first priority in the filter design.
- It is very difficult to test high dimensional filters’ stability.
- There is still no perfect solution to the 2-D filter’s stabilization problem.
1-D recursive filter’s stability test

- 1. Shur-Cohn Stability Criterion.
Jury-Marden Stability Criterion

Consider a filter characterized by the transfer function:

\[ H(z) = \frac{N(z)}{D(z)} \quad N(z) = \sum_{i=0}^{N} a_i z^{-i} \quad D(z) = \sum_{i=0}^{M} b_i z^{-i} \]

<table>
<thead>
<tr>
<th>Row</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( b_0^0 )</td>
</tr>
<tr>
<td>2</td>
<td>( b_M^1 )</td>
</tr>
<tr>
<td>3</td>
<td>( b_0^1 )</td>
</tr>
<tr>
<td>4</td>
<td>( b_{M-1}^2 )</td>
</tr>
<tr>
<td>5</td>
<td>( b_0^2 )</td>
</tr>
<tr>
<td>\vdots</td>
<td>| |</td>
</tr>
<tr>
<td>2M-3</td>
<td>( b_{M-2}^{N-2} )</td>
</tr>
</tbody>
</table>

\[ b_i^k = \begin{bmatrix} b_{k-i}^0 & b_{k-i}^{N-i} \\ b_{k-i}^{-1} & b_{k-i}^{-i} \end{bmatrix} \]

(1) \( D(1) > 0 \)
(2) \( (-1)^N D(-1) > 0 \)
(3) \( b_0^0 > b_N^0 \)
\( |b_0^i| > |b_{N-i}^i| \)  

When \( i > 0 \) and \( i \leq 2M-3 \)
Stabilization of one-dimensional recursive digital filters

- For an unstable recursive filter:

\[ H(z) = \frac{\sum_{i=0}^{M} a_i z^i}{\prod_{i=1}^{K} (z - r_i e^{j\theta}) \prod_{i=K+1}^{M} (z - \frac{1}{r_i} e^{j\theta})} \quad |r_i| < 1 \]

- An all pass filter can stabilize the system:

\[ H(z) = \prod_{i=K+1}^{M} \frac{z - \frac{1}{r_i} e^{i\theta_i}}{z - r_i e^{i\theta_i}} \]
Stability and stabilization of 2-D recursive filters

1) Fundamental theorem of algebra is not applicable to two-variable functions, which means denominator factorization is not always possible.

2) Stabilization is impossible for the high dimensional systems.

3) For 2-D filters even the numerator of the filter’s transfer function can effect the stability of the filter.
Stability Theorem about 2-D filter (1)

Theorem 1: Given that \( D(z_1, z_2) \) is a polynomial in \( z_1 \) and \( z_2 \), a necessary and sufficient condition for the coefficients of the expansion of \( H(z_1, z_2) = 1 / D(z_1, z_2) \) in negative power of \( z_1 \) and \( z_2 \) to converge absolutely, and hence for \( h(m,n) \) to be absolutely summable, is

\[ D(z_1, z_2) \neq 0 \quad \text{for} \quad \bigcap |z_i| \geq 1 \]

The test is done by assigning values to the variable \( z_1 \) and finding the roots of \( D(z_1, z_2) = 0 \) as a function of \( z_2 \). The testing for stability as stated above is very tedious to apply, since it involves mapping an infinite number of points from the \( z_1 \)
Stability Theorem about 2-D filter (2)

- Huang’s two conditions:
- A causal 2-D recursive filter with a z-transfer function

\[ H(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1, z_2)} \]

- Is stable if and only if:

\[ D(z_1, 0) \neq 0, |z_1| \geq 1 \quad \text{Criterion 1} \]

\[ D(z_1, z_2) \neq 0, |z_1| = 1 \cap |z_2| \geq 1 \quad \text{Criterion 2} \]
Modified Jury Stability Test Method---For Criterion 1

**Theorem 3.6.** Let $f(z)$ be the $n$th degree polynomial given by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

(3.74)

where coefficients $a_i$, $i = 0, 1, \ldots, n$ are complex numbers. The roots of $f(z)$ are inside the unit circle if and only if

$$b_0 < 0, \quad c_0 > 0, \quad d_0 > 0, \ldots, g_0 > 0, \ldots, t_0 > 0$$

(3.75)

where $b_0, c_0, \ldots, t_0$ are obtained from the modified Jury’s table formed as follows:

<table>
<thead>
<tr>
<th>$z^0$</th>
<th>$z^1$</th>
<th>$z^2$</th>
<th>$\cdots$</th>
<th>$z^{n-2}$</th>
<th>$z^{n-1}$</th>
<th>$z^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$\cdots$</td>
<td>$a_{n-2}$</td>
<td>$a_{n-1}$</td>
<td>$a_n$</td>
</tr>
<tr>
<td>$a_n^*$</td>
<td>$a_{n-1}^*$</td>
<td>$a_{n-2}^*$</td>
<td>$\cdots$</td>
<td>$a_2^*$</td>
<td>$a_1^*$</td>
<td>$a_0^*$</td>
</tr>
<tr>
<td>$b_0$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$b_{n-2}$</td>
<td>$b_{n-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_{n-1}$</td>
<td>$b_{n-2}$</td>
<td>$b_{n-3}$</td>
<td>$\cdots$</td>
<td>$b_1^*$</td>
<td>$b_0^*$</td>
<td></td>
</tr>
<tr>
<td>$c_0$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$c_{n-2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_{n-2}$</td>
<td>$c_{n-3}$</td>
<td>$\cdots$</td>
<td>$c_0^*$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_0$</td>
<td>$r_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r_1^*$</td>
<td>$r_0^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(3.76)

where

$$b_k = \begin{bmatrix} a_0 & a_{n-k} \\ a_n^* & a_k^* \end{bmatrix} \quad \text{and} \quad c_k = \begin{bmatrix} b_0 & b_{n-1-k} \\ b_{n-1}^* & b_k^* \end{bmatrix}$$

and $a_k^*$ is the complex conjugate of $a_k$. 


Modified Jury Stability Criterion---For Condition 2

We can reconstruct a 2-d function $D(z_1, z_2) = \sum_{i,j=0}^{N} b_{ij} z_1^i z_2^j$ as below:

$$D(z_1, z_2) = \sum_{i=0}^{N} a_i(z_1) z_2^i \quad a_i(z_1) = \sum_{j=0}^{M} b_{ij} z_1^j$$

1. All of the $b_{ij}$ are real, and then $a_j^*(z_1) = a_j(z_1^*)$
2. $Z_1$ is restricted to the boundary $|z|=1$, and thus $|z_1 z_1^*|^l = 1$ for all $l$.
   Then all of the judging functions $(b_j, c_j, d_j, r_j)$ can be expressed as function of $(z_1^* + z_1)$
   
   $$(z_1^* + z_1) = 2X$$
   $$(z_1^{*2} + z_1^2) = 4X^2 - 2$$
   $$(z_1^{*3} + z_1^3) = 8X^3 - 16X$$

3. Since $-1 \leq X \leq 1$, it is easy for us to judge whether:

   $$b_0(X) < 0 \quad c_0(X) > 0 \quad r_0(X) > 0$$
Conventional stabilization of 2D recursive filter

- NO PERFECT SOLUTION.
Proposed computer aided algorithm

- Computer aided analysis of Huang’s theorem 2

\[ D(z_1, z_2) \neq 0, \quad |z_1| = 1 \cap |z_2| \geq 1 \]

\[ z_1 = e^{j\theta_i}, \quad \theta_i \in [-\pi, \pi], \quad i \in [1, N] \]

\( N \) is the total number of the sample angles
Computer aided analysis of Huang’s criterion 2

For the denominator function \( D(z_1, z_2) = 0 \)

- Since \( |z_1| = 1 \), we substitute \( z_1 = e^{j \theta_i} \)

\[
D(z_2, \theta_i) = 0 \\
D(z_1, z_2) = \sum_{j=0}^{N} a_j(\theta_i)z_2^i
\]

Here \( \theta_i \) is a constant to the function \( z_2 \)

With the aid of computer, it’s easy for us to factorize above function as below:

\[
\prod_{j=1}^{N} (z_2 - p_j(\theta_i)) = 0 \quad \text{N is the order of the polynomial of } Z_2
\]
Tracking the poles with the aid of computer

Here $\Delta \theta$ is a small value. So when we change the angle from 0 to $\pi$, then the N complex poles will move from the start to end.

\[
\begin{align*}
\theta_{0} &= \theta_{0} \\
* \ p1 \\
* \ p2 \\
* \ p3 \\
\theta_{2} &= \theta_{0} + 2\Delta \theta \\
* \ p1 \\
* \ p2 \\
* \ p3 \\
\theta_{1} &= \theta_{0} + \Delta \theta \\
* \ p1 \\
* \ p2 \\
* \ p3 \\
\theta_{3} &= \theta_{0} + 3\Delta \theta \\
* \ p1 \\
* \ p2 \\
* \ p3
\end{align*}
\]
A demonstration of the complex value $Z_2$ changing with the $\theta$.
Feasibility of tracking the poles (1)

For $z_{2j} = p_j(\theta_i) : \quad \theta \in (0, \pi)$

1. $Z_2$ is finite. Since: $D(z_1, z_2) = \sum_{i=0}^{N} a_i(z_1)z_2^i = 0$ while $|Z_1| = 1$ obviously $a_i(z_1) < \infty$

2. For any $\theta \in (0, \pi)$, there is a an available complex value $Z$

3. The $p_j(\theta_i)$ is continuous function. For any $\theta \in (0, \pi)$

$$p_j(\theta_i^+) = p_j(\theta_i^-)$$
Feasibility of tracking the poles(2)

- If \[ D(z_1, z_2) = (z_1 - p_1)^k D'(z_1, z_2) = 0 \quad |p_1| = 1 \] Then any \( z_2 \) can be solution to the equation.

- If \[ D(z_1, z_2) = (z_1 - p_1)^k D'(z_1, z_2) + c = 0 \quad |p_1| = 1 \] while \( c \) is constant and isn’t zero. There is no solution to \( z_2 \).

It is always possible to decide whether there are factors of \( (z_1 - p_1) \) or \( (z_2 - p_2) \) in the function \( D(z_1, z_2) \)
If \( D(z_{1i}, z_{2i}) = 0 \) \( z_{1(i+1)} = z_{1(i)} + \Delta z_1 \) \( z_{2(i+1)} = z_{2(i)} + \Delta z_2 \)

\[
D(z_{1(i+1)}, z_{2(i+1)}) = 0
\]

\[
D(z_{1(i+1)}, z_{2(i+1)}) = D(z_{1i}, z_{2i}) + \frac{\partial D(z_1, z_2)}{\partial z_1} \times \Delta z_1 + \frac{\partial D(z_1, z_2)}{\partial z_2} \times \Delta z_2
\]

\[
= \frac{\partial D(z_1, z_2)}{\partial z_1} \times \Delta z_1 + \frac{\partial D(z_1, z_2)}{\partial z_2} \times \Delta z_2
\]

\[
\frac{\partial D(z_1, z_2)}{\partial z_1} \times \Delta z_1 = -\frac{\partial D(z_1, z_2)}{\partial z_2} \times \Delta z_2
\]

\[
\Delta z_2 = -\Delta z_1 \times \left( \frac{\partial D(z_1, z_2)}{\partial z_1} \right) / \left( \frac{\partial D(z_1, z_2)}{\partial z_2} \right)
\]
Feasibility of tracking the poles (4)

\[ \Delta z_2 = -\Delta z_1 \times \left( \frac{\partial D(z_1, z_2)}{\partial z_1} \right) \left( \frac{\partial D(z_1, z_2)}{\partial z_2} \right) \]

\[ \Delta z_1 = \varepsilon \]

Case 1: \[ \frac{\partial D(z_1, z_2)}{\partial z_1} \neq 0 \quad \frac{\partial D(z_1, z_2)}{\partial z_2} \neq 0 \]
\[ \Delta z_2 = \varepsilon \]

Case 2: \[ \frac{\partial D(z_1, z_2)}{\partial z_1} \neq 0 \quad \frac{\partial D(z_1, z_2)}{\partial z_2} = 0 \]
\[ D(z_1, z_2) = (z_1 - z_{1i}) \quad D'(z_1, z_2) + c = 0 \]

Case 3: \[ \frac{\partial D(z_1, z_2)}{\partial z_1} = 0 \quad \frac{\partial D(z_1, z_2)}{\partial z_2} \neq 0 \]
\[ \Delta z_2 = 0 \]

Case 4: \[ \frac{\partial D(z_1, z_2)}{\partial z_1} = 0 \quad \frac{\partial D(z_1, z_2)}{\partial z_2} = 0 \]
\[ D(z_1, z_2) = D_1(z_1, z_2)^k D_2(z_1, z_2) \]

\[ z_1, z_2 \] are the zeros of the \( D_1(z_1, z_2) \), this is easy to be identified by our method since the poles are overlapped.
Approximation of the poles’ functions

- We can get $N$ functions about the magnitudes of the poles, the variable is the angle the $z_1$ rotating around the unit circle.

$$\theta \in (0, \pi)$$

$$A_j = |z_{2j}| = |p_j(\theta)| = \sqrt{\text{real}(p_j(\theta))^2 + \text{image}(p_j(\theta))^2} = \text{Function}_j(\theta)$$

$$j \in (1, N) \quad N \text{ is the number of the poles in the } z_2 \text{ function}$$

We sample $M$ values from 0 to pi, then we can get the $M$ $A_{jm}$. 
Implementation of Taylor series (1)

- According to Taylor function we can approximate the function into polynomial function. We have n individual such functions.

\[ A = Function(\theta) = \sum_{i=0}^{M} a_i \times \theta^i \]

For each function we have M sample \( \theta_m \)

\[
\begin{bmatrix}
\theta_1^m & \theta_1^{m-1} & 1
\end{bmatrix} \begin{bmatrix} a_0 \end{bmatrix} = \begin{bmatrix} A_1 \end{bmatrix}
\]

\[
\begin{bmatrix}
\theta_2^m & \theta_2^{m-1} & 1
\end{bmatrix} \begin{bmatrix} a_1 \end{bmatrix} = \begin{bmatrix} A_2 \end{bmatrix}
\]

\[
\begin{bmatrix}
\theta_3^m & \theta_3^{m-1} & 1
\end{bmatrix} \begin{bmatrix} a_2 \end{bmatrix} = \begin{bmatrix} A_3 \end{bmatrix}
\]

\[
\begin{bmatrix}
\theta_{m+1}^m & \theta_{m+1}^{m-1} & 1
\end{bmatrix} \begin{bmatrix} a_m \end{bmatrix} = \begin{bmatrix} A_{m+1} \end{bmatrix}
\]
Implementation of Taylor series (2)

Obviously it’s easy for us to get the $a_i$ in the below function

$$A = \text{Function} (\theta) = \sum_{i=0}^{M} a_i \times \theta^i \quad \theta \in (0, \pi)$$

Then it is easy for us to get the range of $A$ when the angle has limited area. The minimum or maximum happen in below two cases:

1) When $\theta$ is at the end of the area.
2) When $\frac{\partial A}{\partial \theta} = 0$
Procedure of the stability test

1) Convert a 2-D polynomial into a matrix

\[ D(z_1, z_2) = \sum a_{ij} z_1^i z_2^j \]

Then we can get a matrix

\[ A = \begin{bmatrix} a_{ij} \end{bmatrix} \]

\[ A = \begin{bmatrix} -0.1 & -0.01 & -0.2 \\ -0.01 & -0.24 & -0.1 \\ 0.02 & 0.3 & 1 \end{bmatrix} \]
The simulation result

Curve of the pole 1

Curve of the pole 2
Stabilization of two-dimensional recursive digital filters (1)

- For an 2D recursive filter: \( H(z_1, z_2) = \frac{N(z_1, z_2)}{D(z_1, z_2)} \)
- Can be converted into function like:

\[
H(z_1, z_2) = \frac{N(z_1, z_2)}{\prod_{i=0}^{n} (z_2 - p_i(z_1))}
\]

The functions of \( p_i(z_1) \) can be use the sampling the \( z_1 \) around the unit circle, with the same method as before we can get a Taylor function for each pole.

\[
p_i(z_1) = \sum_{i=0}^{M} a_i \times z_1^i
\]
Stabilization of two-dimensional recursive digital filters (2)

- For different poles the corresponding unstable angles can be found and can be modify the coefficient to get the new response to guarantee every pole is stable.
Illustration of modified amplitude response.
Future job

1. When the number of sample is high, it’s difficult to compute (at least for the matlab) to get the accurate inverse matrix.

2. I haven’t find the error range of the Taylor series.

3. Do some more simulations about the stabilization.